Derivative and primitive of compact support splines. The case of convolution power.

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Abstract

As potências por convolução da função característica do intervalo $[0, 1]$ têm uma propriedade muito interessante que torna fácil o cálculo da próxima potência. Alguma propriedades e exemplos são o objetivo deste artigo.

palavras chave: função característica, potências por convolução, um teorema das potências por convolução.

Convolution power of characteristic function of $[0, 1]$ have a nice property which makes easy to obtain the next power. Some properties and examples are the point with this paper.

keywords: characteristic function, convolution power, a theorem of convolution power.

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1 The project

In the second section I will describe the fundamental property of convolution power of the characteristic function of the interval $[0, 1]$ and present a proof.

In the third section I will give an example showing the calculus from the second power to the third power. In the fourth section I will show how I have used `gnuplot` as a checking machine for the calculation process. In the last section I will display the graphic of the sixth convolution power of $\chi_{[0,1]}$ and its four continuous derivatives because $\chi^6_{[0,1]}$ is a 4-splines.

2 Fundamental property of convolution power of characteristic function.

Theorem 1 (The fundamental property)**.** *of convolution power of characteristic function.*
Let $f_1 = \chi = \chi_{[0,1]}$ and $f_n = \chi = \chi^n_{[0,1]}$, then

$$\frac{df_n(x)}{dx} = f_{n-1}(x) - f_{n-1}(x-1) \tag{1}$$

Proof. It is well known that for convolution it is true for two integrable functions that

$$\frac{d}{dx} f * g = \frac{df}{dx} * g = f * \frac{dg}{dx}; \tag{2}$$

where the derivative is to be taken in the distribution sense when needed. Then if follows

$$f_n = \chi_{[0,1]} * f_{n-1} = f_1 * f_{n-1}; \tag{3}$$

$$\frac{df_n(x)}{dx} = \frac{df_1}{dx} * f_{n-1} = \frac{d\chi_{[0,1]}}{dx} * f_{n-1}(x) = (\delta_0 - \delta_1) * f_{n-1}(x) = f_n(x) - f_n(x-1); \tag{4}$$

in the last equation I have used derivation in the sense of distribution and Dirac measure pops up in the calculus of the derivative of f_1. □

3 The next power

The second convolution power of $\chi_{[0,1]}$ is a continuous function, but it is not derivable, or it has non-continuous derivative, the derivatives of the two straight lines segments that form the second power by convolution producing the triangle function whose support is the interval $[0, 2]$ with the value 1 at the midpoint of the support. Here it may be observed a fundamental property of the convolution product which is the preservation of the integral when one of the factors has integral 1.

Following the notation already suggested in the first section, $\chi_{[0,1]} = f_1$, the triangle function is f_2 being its derivative the difference $f_1(x) - f_1(x-1)$, the difference between f_1 and its translation of a unit, a discontinuous function.

By the theorem 1 the derivative of the third power, f_3 is the difference of translations f_2. In the following equations you can see all the calculations that led me to the third power of the characteristic function.

$$\frac{df_3(x)}{dx} = f_2(x) - f_2(x-1); \tag{5}$$

$$f_2(x) = \begin{cases} x \leq 0 \quad \Rightarrow \quad 0; \\ x \in [0,1] \quad \Rightarrow \quad x; \\ x \in [1,2] \quad \Rightarrow \quad 2-x; \\ x \geq 2 \quad \Rightarrow \quad 0; \end{cases} \tag{6}$$

$$f_2(x) - f_2(x-1) = \begin{cases} x \leq 0 \quad \Rightarrow \quad 0; \\ x \in [0,1] \quad \Rightarrow \quad x; \\ x \in [1,2] \quad \Rightarrow \quad 2-x-(x-1) = 3-2x; \\ x \geq [2,3] \quad \Rightarrow \quad -(2-(x-1)) = -2+(x-1) = -3+x; \\ x \geq 3 \quad \Rightarrow \quad 0; \end{cases} \tag{7}$$

$$df_3(x) = \begin{cases} x \leq 0 \quad \Rightarrow \quad 0; \\ x \in [0,1] \quad \Rightarrow \quad x; \\ x \in [1,2] \quad \Rightarrow \quad 3-2x; \\ x \geq [2,3] \quad \Rightarrow \quad -3+x; \\ x \geq 3 \quad \Rightarrow \quad 0; \end{cases} \tag{8}$$

$$f3(x) = \begin{cases} x \leq 0 \quad \Rightarrow \quad 0; \\ x \in [0,1] \quad \Rightarrow \quad x^2/2; \\ x \in [1,2] \quad \Rightarrow \quad 1/2-(3-1)+3x-x^2 = -1.5+3x-x^2; \\ x \in [2,3] \quad \Rightarrow \quad 1/2-(-4)-3x+x^2/2 = 4.5-3x+x^2/2.0; \\ x \geq 3 \quad \Rightarrow \quad 0; \end{cases} \tag{9}$$

$$f3(x) = \begin{cases} x \leq 0 \quad \Rightarrow \quad 0; \\ x \in [0,1] \quad \Rightarrow \quad x^2/2; \\ x \in [1,2] \quad \Rightarrow \quad -1.5+3x-x^2; \\ x \in [2,3] \quad \Rightarrow \quad 4.5-3x+x^2/2.0; \\ x \geq 3 \quad \Rightarrow \quad 0; \end{cases} \tag{10}$$

The graphic, made with these equations translated to `gnuplot` is at the figure (1), 2.

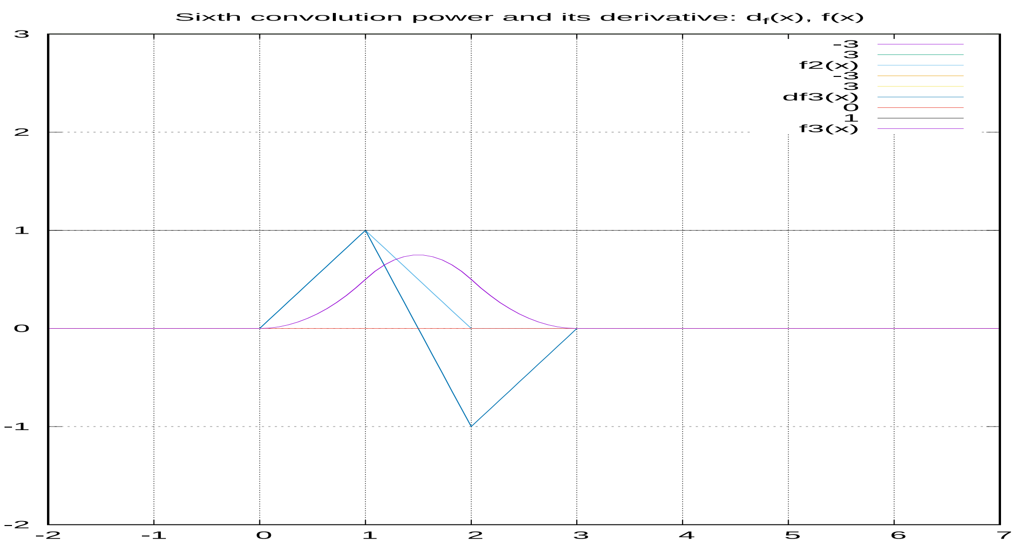

Figure 1: second and third power

4 Verifying the correction computationally

It is a big work to calculate $f2(x) - f2(x-1) = \frac{df3(x)}{dx}$ but it's still simpler than the direct calculation of $\chi * f2 = f3$. The methodology I have used to commit fewer errors was to produce the graph with gnuplot at each line of the equation and thus immediately correct the error in the line in progress without accumulating errors. It was extremely useful in calculating the sixth power which is a system of equations with 7 lines and an error committed in one line accumulates for all following ones. Here is the calculation done inside a terminal of gnuplot to get $f6$ from the already made calculation of $f5$

```
f5(x) = (x<=0) ?0:(x<=1)?power(x,4)/24.0:\
(x<=2)?-power(x,4)/6.0 + 5*power(x,3)/6.0 +\
-5.0*power(x,2)/4.0 +5*x/6.0 -5/24.0:\
(x<=3)?power(x,4)/4.0 - 5*power(x,3)/2.0+\
35*power(x,2)/4.0 - 75*x/6.0 + 155/24.0:\
(x<=4)?-power(x,4)/6.0 +5*power(x,3)/2.0 +\
- 55*power(x,2)/4.0 +195*x/6.0 -655/24.0:\
(x<=5)?power(x,4)/24.0 - 5*power(x,3)/6.0 +\
25*power(x,2)/4.0 - 125*x/6.0 + 625/24.0:0;
df6(x) = f5(x)-f5(x-1);
f6(x) = (x<=0)? 0:\
        (x<=1)? power(x,5)/120.0:\
        (x<=2)? 1/20.0 -x/4.0+\
                + power(x,2)/2.0 - power(x,3)/2.0+\
                + power(x,4)/4.0 - power(x,5)/24.0:\
        (x<=3)? -237/60.0 + 117*x/12.0 - 19*power(x,2)/2.0 +\
                9*power(x,3)/2.0 - power(x,4) + power(x,5)/12.0:\
        (x<=4)? 731/20.0 - 231*x/4.0 + 71*power(x,2)/2.0+\
 - 21*power(x,3)/2.0 +\
                3*power(x,4)/2.0 - power(x,5)/12.0:\
        (x<=5)? -5487/60.0  +\
                409*x/4.0 - 89*power(x,2)/2.0 +\
                19*power(x,3)/2.0 - power(x,4) +\
                power(x,5)/24.0:\
(x<=6)?  324/5.0 -54*x + 18*power(x,2) +\
-3*power(x,3) + power(x,4)/4.0 -power(x,5)/120.0:0;
```

If you are not used with gnuplot, let me tell you that there are two ways to to define if/else and the one being used above is the compact form of the C language

```
if (A) B; else C;  is equivalent to  (A)?B:C
```

gnuplot accepts that a line be written along multiple lines using the "\" to eliminate the end of line. So I can write the compact if/else over several lines, in a more legible faction. Before moving on to the next inequality, I would have finished the else part,":', either with ":0' or ":1'. In case of the derivative using ":0', because the derivatives of the convolution powers are always compact support, and ":1', when calculating primitives because in this casa I didn't want anything greater than 1. This gave me the chance to quickly see where there were errors.

Latest version of gnuplot has a non comp version of if/else.

Another technique I've also used, and I do commonly use, is asking gnuplot the graphic f and its tangent line at some points to test if I have calculated correctly the derivative f'.

5 Compact support splines and its derivatives

The figure (2), at page 4, shows the graphics of the sixth convolution power of $\chi[0, 1]$ along with their four continuous derivatives, $f6$ has four continuous derivatives.

the graphics were made with `gnuplot` with the following 4 command lines.

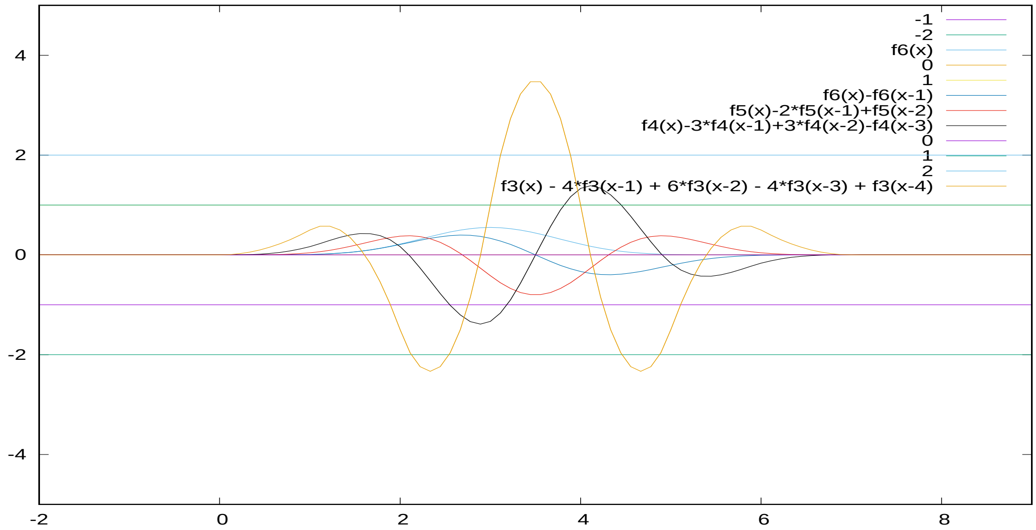

Figure 2:

```
set terminal postscript eps enhanced color
set output 'DerivadasPrimitivaSplinesCompactSupp.eps'
set xrange [-2:9]; set yrange [-5:5];
plot -1,-2,\
        f6(x),\
        f6(x)-f6(x-1),\
        f5(x) - 2*f5(x-1) + f5(x-2),\
        f4(x) - 3*f4(x-1) + 3*f4(x-2) - f4(x-3),\
        f3(x) - 4*f3(x-1) + 6*f3(x-2) - 4*f3(x-3) + f3(x-4),\
        0,1,2
```

and notice what I have already said, the back slash, “\”, eliminate the end of line which allowed me to write the last line more legibly, separating the different functions, each in one “apparent line”, which will be read by `gnuplot` as a single line.

This paper contains the evolution of the results obtained in [2] where the results were computational leading to a program to calculate any convolution power of the characteristic function. In this step I started again from the second power to calculate the nth power, using the Theorem 1. Now the formulas obtained are the precise algebraic expressions of the pieces of polynomials involved. It is not possible, with this methodology, to obtain any convolution power of the characteristic function, but, when calculated f_n, it's easy to get f_{n+1} with the methodology described here.

Some of the programs I have used in this paper can be obtained from [3] or requested from the author if not found in the indicated citation. The programs are experimental and it is need a habit with `gnuplot` to eliminate comments which are hiding certain operations. Most of the comments hide the tests that I referred to in the previous section. You should look for

```
Derivada.gnuplot
```

```
 convolution\_power.gnuplot
 bibioteca.gnuplot
 Very old but this was the begining in 2011,

potencia\_convolucao01_zero_um.gnuplot
```

I hope you can survive!

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References

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